Finite Math - Fall 2016

Lecture Notes - 9/14/2016

Section 3.2 - Compound and Continuous Compound Interest

We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

Example 1. *How long will it take* $\$10,000$ *to grow to* $\$25,000$ *if it is invested at* 8% *compounded quarterly?*

Solution. *Here, we have* $P = \$10,000$, $A = \$25,000$, $r = 0.08$, $i = \frac{0.08}{4} = 0.02$, *thus the model gives us*

$$\$25,000 = \$10,000(1 + 0.02)^n$$

and so we can solve for the number of compounding periods required.

$$\begin{aligned} \$25,000 &= \$10,000(1+0.02)^n \\ 2.5 &= 1.02^n \\ \ln 2.5 &= n \ln 1.02 \\ n &= \frac{\ln 2.5}{\ln 1.02} = 46.27 \end{aligned}$$

So, this means we need 47 *quarters to achieve* $\$25,000$, *or* 11 *years and* 3 *quarters.*

Example 2. *How long will it take money to triple if it is invested at (a)* 5% *compounded daily? (b)* 6% *compounded continuously?*

Solution.

(a) $8,021$ *days (about* 21.975 *years)*

(b) 18.310 *years*

We can also look to figure out the desired interest rate if we know the present value, the length of time, and the desired future value.

Example 3. *The Russell Index tracks the average performance of various groups of stocks. On average, a* $\$10,000$ *investment in mid-cap growth funds over a* 10*-year period would have grown to* $\$63,000$. *(A mid-cap fund is a type of stock fund that invests in mid-sized companies. See Investopedia for more information.) What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously. Express answers as a percentage, rounded to three decimal places.*

Solution. *Here, we have* $P = \$10,000$, $t = 10$, *and* $A = \$63,000$ *as given values. We are looking for the rate* r *in both cases.*

(a) *If interest is compounded annually, then the interest per compounding period is* $i = \frac{r}{1} = r$ *and the number of compounding periods is* $n = 1(10) = 10$. *So the formula gives us*

$$\$63,000 = \$10,000(1+r)^{10}.$$

We first need to isolate $(1+r)^{10}$, *then take the* 10^{th} *root of both sides (recall that* $x^{10} = \sqrt[10]{x}$*)*

$$\begin{aligned} 63,000 &= 10,000(1+r)^{10} \\ 6.3 &= (1+r)^{10} \\ \sqrt[10]{6.3} &= 1+r \\ r &= \sqrt[10]{6.3} - 1 = 1.20208 - 1 = 0.20208 = 20.208\% \end{aligned}$$

(b) *If interest is compounded continuously, then*

$$\$63,000 = \$10,000e^{r(10)} = \$10,000e^{10r}$$

we first isolate the exponential term, then use $\ln$ *to get the exponent out:*

$$\begin{aligned} 63,000 &= 10,000e^{10r} \\ 6.3 &= e^{10r} \\ \ln 6.3 &= 10r \\ r &= \frac{\ln 6.3}{10} = 0.18405 = 18.405\% \end{aligned}$$

Example 4. *A promissory note will pay* $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate compounded continuously would you earn?*

Solution. 9.66%

Annual Percentage Yield. Suppose we are looking at various certificates of deposit (CDs) from different banks and we've come across the following ones

Bank	Rate	Compounded
Advanta	4.93%	monthly
DeepGreen	4.95%	daily
Charter One	4.97%	quarterly
Liberty	4.94%	continuously

How can we tell which has the largest return on our investment? Looking strictly at percentages would suggest Charter One, while looking a compounding periods might suggest Liberty, but, in fact, it is neither of these. For example, if we purchased a $1,000 CD from each bank, our return in each case would be

Bank	Return
Advanta	$1,050.43
DeepGreen	$1,050.74
Charter One	$1,050.63
Liberty	$1,050.64

So we see that DeepGreen is the best value. We can then ask ourselves if we can come up with a nice number to compare all of these by, a sort of standard reference. This standardized number is called the *Annual Percentage Yield*. What this number does is tell you how much your investment will grow by at the end of 1 year. In a sense, it is the *effective interest rate*. How do we get this then? We use the following idea

$$\begin{array}{c}\text{amount at}\\ \text{simple interest}\\ \text{after 1 year}\end{array} = \begin{array}{c}\text{amount at}\\ \text{compound interest}\\ \text{after 1 year}\end{array}$$

Solving for the simple interest rate on the left will tell us, effectively, how much interest is made in a year. We will call that simple interest rate the *Annual Percentage Yield*, or APY. For compound interest:

$$\begin{aligned} P(1+APY) &= P\left(1+\frac{r}{m}\right)^m \\ 1+APY &= \left(1+\frac{r}{m}\right)^m \\ APY &= \left(1+\frac{r}{m}\right)^m - 1 \end{aligned}$$

and in the continuously compounded case:

$$\begin{aligned} P(1+APY) &= Pe^r \\ 1+APY &= e^r \\ APY &= e^r - 1 \end{aligned}$$

Definition 1 (Annual Percentage Yield)**.** *If a principal is invested at the annual (nominal) rate r compounded m times a year, then the annual percentage yield is*

$$APY = \left(1+\frac{r}{m}\right)^m - 1$$

If a principal is invested at the annual (nominal) rate r compounded continuously, then the annual percentage yield is

$$APY = e^r - 1$$

Example 5. *Southern Pacific Bank offered a 1-year CD that paid 4.8% compounded daily and Washington Savings Bank offered one that paid 4.85% compounded quarterly. Find the APY for each CD. Which has a higher return?*

Solution. *For Southern Pacific Bank, the APY is*

$$APY = \left(1+\frac{0.048}{365}\right)^{365} - 1 = 1.04917 - 1 = .04917 = 4.917\%$$

For Washington Savings Bank, the APY is

$$APY = \left(1 + \frac{0.0485}{4}\right)^4 - 1 = 1.04939 - 1 = .04939 = 4.939\%$$

So, since Washington Savings has the higher APY, the CD with them has the higher return.

Example 6. *An online bank listed a* 1*-year CD that earns* 1.25% *compounded monthly. Find the APY as a percentage, rounded to three decimal places.*

Solution. 1.257%

You can also solve this backwards and find the nominal rate from the APY:

Example 7. *What is the annual nominal rate compounded monthly for a CD that has an annual percentage yield of* 5.9%*?*

Solution. 5.75%

Section 3.3 - Future Value of an Annuity; Sinking Funds

Annuities. At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal in this section will be to find the future value of an annuity.

Example 8. *Suppose you decide to deposit* \$100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

Solution. *We can visualize the value of each of those* \$100 *deposits in a table.*

Deposit	**Term Deposited**	**Number of times Compounded**	**Future Value**
\$100	1	5	$\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$
\$100	2	4	$\$100\left(1+\frac{0.06}{2}\right)^4 = \$100(1.03)^4$
\$100	3	3	$\$100\left(1+\frac{0.06}{2}\right)^3 = \$100(1.03)^3$
\$100	4	2	$\$100\left(1+\frac{0.06}{2}\right)^2 = \$100(1.03)^2$
\$100	5	1	$\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$
\$100	6	0	$\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$

So adding up the future values of all these will give us the amount of money in the account

$$B = \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 + \$100(1.03)^2 + \$100(1.03) + \$100 = \$646.84$$

There is actually a nice trick to adding this up. Begin by writing the same list, but let's multiply it by 1.03

$$1.03B = \$100(1.03)^6 + \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 + \$100(1.03)^2 + \$100(1.03)$$

then we compute $1.03B - B$:

$$\begin{array}{rcccccccccccc} 1.03B & = & \$100(1.03)^6 & + & \cancel{\$100(1.03)^5} & + & \cancel{\$100(1.03)^4} & + & \cancel{\$100(1.03)^3} & + & \cancel{\$100(1.03)^2} & + & \cancel{\$100(1.03)} & & \\ -B & = & & - & \cancel{\$100(1.03)^5} & - & \cancel{\$100(1.03)^4} & - & \cancel{\$100(1.03)^3} & - & \cancel{\$100(1.03)^2} & - & \cancel{\$100(1.03)} & - & \$100 \end{array}$$

So we arrive at

$$1.03B - B = .03B = \$100(1.03)^6 - \$100 = \$100\left((1 + 0.03)^6 - 1\right)$$

Solving for B gives

$$B = \$100\frac{(1 + 0.03)^6 - 1}{0.03} = \$646.84.$$

This gives rise to the following formula

Definition 2 (Future Value of an Ordinary Annuity)**.**

$$FV = PMT\frac{(1 + i)^n - 1}{i}$$

where

$$\begin{array}{rcl} FV & = & \textit{future value} \\ PMT & = & \textit{periodic payment} \\ i & = & \textit{rate per period} \\ n & = & \textit{number of payments (periods)} \end{array}$$

Note that the payments are made at the end of each period.

In the above formula, $i = \frac{r}{m}$, where r is the interest rate (as a decimal) and m is the number compounding periods per year and $n = mt$ where t is the length of time of the investment. We can rewrite the formula with r and m instead of i

$$FV = PMT\frac{\left(1 + \frac{r}{m}\right)^n - 1}{r/m}$$

Example 9. *What is the value of an annuity at the end of* 10 *years if* \$1,000 *is deposited every* 3 *months into an account earning* 8% *compounded quarterly. How much of this value is interest?*

Solution. *In this problem,* $PMT = \$1,000$*,* $r = 0.08$*,* $m = 4$*, and* $n = 4(10) = 40$*. So* $\frac{r}{m} = \frac{0.08}{4} = 0.02$ *and the future value is*

$$FV = \$1,000\frac{\left(1 + \frac{0.08}{4}\right)^{40} - 1}{0.08/4} = \$1,000\frac{(1.02)^{40} - 1}{0.02} = \$60,401.98$$

To figure out how much is interest, we simply figure out how much money we put into the account, then subtract that from the future value. Since we made 40 *payments of* \$1,000*, we invested* $40(\$1,000) = \$40,000$*. Thus the interest is*

$$I = \$60,401.98 - \$40,000 = \$20,401.98.$$

Example 10. *If* $\$1,000$ *is deposited at the end of each year for* 5 *years into an ordinary annuity earning* 8.32% *compounded annually, what will be the value of the annuity at the end of the* 5 *years?*

Solution. $\$5,904.15$